Minimum Weight Triangulation

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Problem 1 from The Open Problems Project http://maven.smith.edu/~orourke/TOPP/P1.html#gj-cigtn-79
The MWT

• A minimum weight triangulation of P is a triangulation of P which has a minimum sum of the Euclidean lengths of its edges.

• Application:
  – Optimal Triangulation
  – Stock cutting,
  – finite element analysis,
  – terrain modeling,
  – numerical approximation
Example Triangulations

FIG. 1. A planar point set and different ways to triangulate it. The greedy triangulation (a) is constructed incrementally, always adding the shortest possible edge. In this example, it is shorter than the Delaunay triangulation (b), which avoids skinny triangles. A minimum weight triangulation for the point set is shown in (c). The length $L$ of the MWT can decrease when additional Steiner points are allowed (d).

Figure taken from:
The Question

• *Can a minimum weight triangulation of a planar point set---one minimizing the total edge length--be found in polynomial time?*
Approaches

• Minimum Weight Steiner Triangulation:
  – *David Eppstein*: shows that by using a Quadtree and $O(n)$ Steiner points a MWST can be approximated. Additionally the MWT has weight of $\Theta(n)$ times that of the MWST.

  – Running time $O(n \log n + k)$, where k is the number of Steiner points.
A Quadtree

An example of a region partitioned by a Quadtree
Approaches

• Minimum Weight Steiner Triangulation:
  – The Quadtree:
    • partitions the space which has the input set of points
    • each point is alone in a square

  – Steiner points:
    • add a Steiner point at each location where the convex hull of
      the points crosses a line segment of the quadtree.
    • include as Steiner points all corners of quadtree squares
Approaches

• Greedy Approximations:
  – *Christos Levcopoulos and Drago Krznaric*: show a quasi-greedy approach that approximates the optimal by a tight factor of $\Omega(\sqrt{n})$. Running time is $O(n \log n)$.

  – *Matthew T. Dickerson, Scott A. McElfresh, and Mark H. Montague*: give $O(n^2)$ running times for several algorithms for general cases.
Polynomial-Time Solvable Cases

• For an \( n \)-vertex convex polygon, the problem can be solved in \( O(n^3) \) using dynamic programming.

• When a given point set lies on a constant number of nested convex hulls.
Solution

- **Wolfgang Mulzer and Günter Rote `06:**
  - Showed that the MWT is an NP-Hard problem.
  
  - Used a polynomial time reduction from Positive Planar 1-in-3 SAT problem to MWT.
  
  - 1-in-3 SAT is a known NP-Hard problem.
Further Work

• It is not known whether the MWT problem is in NP, since it is not known how to compare sums of Euclidean lengths in polynomial time.