CH. 8
PRIORITY QUEUES AND HEAPS

ACKNOWLEDGEMENT: THESE SLIDES ARE ADAPTED FROM SLIDES PROVIDED WITH DATA STRUCTURES AND ALGORITHMS IN C++, GOODRICH, TAMASSIA AND MOUNT (WILEY 2004) AND SLIDES FROM NANCY M. AMATO AND JORY DENNY
OUTLINE AND READING

- PriorityQueue ADT (Ch. 8.1)
  - Total order relation (Ch. 8.1.1)
  - Comparator ADT (Ch. 8.1.2)
  - Implementing a PQ with a list (Ch. 8.2)
- Heaps (Ch. 8.3)
  - Complete Binary Trees (Ch. 8.3.2)
  - Implementing a PQ with a heap (Ch. 8.3.3)
- Sorting with Priority Queue (Ch. 8.1.5)
  - List based PQ: Selection-sort and Insertion Sort (Ch. 8.2.2)
  - Heaps: Heapsort (Ch. 8.3.5)
PRIORITY QUEUES

- Stores a collection of elements each with an associated “key” value
  - Can insert as many elements in any order
  - Only can inspect and remove a single element – the minimum (or maximum depending) element

- Applications
  - Standby Flyers
  - Auctions
  - Stock market
• Keys in a priority queue can be arbitrary objects on which an order is defined, e.g., integers
• Two distinct items in a priority queue can have the same key

• Mathematical concept of total order relation $\leq$
  • Reflexive property:
    \[ k \leq k \]
  • Antisymmetric property:
    if $k_1 \leq k_2$ and $k_2 \leq k_1$, then $k_1 = k_2$
  • Transitive property:
    if $k_1 \leq k_2$ and $k_2 \leq k_3$ then $k_1 \leq k_3$
A comparator encapsulates the action of comparing two objects according to a given total order relation.

A generic priority queue uses a comparator as a template argument, to define the comparison function ($\leq$).

The comparator is external to the keys being compared. Thus, the same objects can be sorted in different ways by using different comparators.

When the priority queue needs to compare two keys, it uses its comparator.
PRIORITY QUEUE ADT

- A priority queue stores a collection of items each with an associated “key” value
- Main methods
  - insert(e) – inserts an element e
  - removeMin() – removes the item with the smallest key
  - min() – return an element with the smallest key
  - size(), empty()
LIST-BASED PRIORITY QUEUE

Unsorted list implementation
• Store the items of the priority queue in a list, in arbitrary order

```
4 5 2 3 1
```

• Performance:
  • `insert(e)` takes $O(1)$ time since we can insert the item at the beginning or end of the list
  • `removeMin()` and `min()` take $O(n)$ time since we have to traverse the entire sequence to find the smallest key

Sorted list implementation
• Store the items of the priority queue in a list, sorted by key

```
1 2 3 4 5
```

• Performance:
  • `insert(e)` takes $O(n)$ time since we have to find the place where to insert the item
  • `removeMin()` and `min()` take $O(1)$ time since the smallest key is at the beginning of the list
HEAPS
WHAT IS A HEAP?

- A heap is a binary tree storing keys at its internal nodes and satisfying the following properties:
  - **Heap-Order**: for every node $v$ other than the root, $\text{key}(v) \geq \text{key}(v.\text{parent})$
  - **Complete Binary Tree**: let $h$ be the height of the heap
    - for $i = 0 \ldots h - 1$, there are $2^i$ nodes on level $i$
    - at level $h$, nodes are filled from left to right
- Can be used to store a priority queue efficiently
**Theorem:** A heap storing $n$ keys has height $O(\log n)$

**Proof:** (we apply the complete binary tree property)

- Let $h$ be the height of a heap storing $h$ keys
- Since there are $2^i$ keys at level $i = 0 \ldots h - 1$ and at least one key on level $h$, we have
  $$n \geq 1 + 2 + 4 + \cdots + 2^{h-1} + 1 = (2^h - 1) + 1 = 2^h$$
- Level $h$ has at most $2^h$ nodes: $n \leq 2^{h+1} - 1$
- Thus, $\log(n + 1) - 1 \leq h \leq \log n$
Let H be a heap with 7 distinct elements (1,2,3,4,5,6, and 7). Is it possible that a preorder traversal visits the elements in sorted order? What about an inorder traversal or a postorder traversal? In each case, either show such a heap or prove that none exists.
insert(e) consists of three steps
- Find the insertion node z (the new last node)
- Store e at z and expand z into an internal node
- Restore the heap-order property (discussed next)
UPHEAP

- After the insertion of a new element $e$, the heap-order property may be violated.
- \textbf{Up-heap bubbling} restores the heap-order property by swapping $e$ along an upward path from the insertion node.
- Upheap terminates when $e$ reaches the root or a node whose parent has a key smaller than or equal to $\text{key}(e)$.
- Since a heap has height $O(\log n)$, upheap runs in $O(\log n)$ time.
• `removeMin()` corresponds to the removal of the root from the heap

• The removal algorithm consists of three steps
  • Replace the root with the element of the last node \( w \)
  • Compress \( w \) and its children into a leaf
  • Restore the heap-order property (discussed next)
• After replacing the root element of the last node, the heap-order property may be violated.
• **Down-heap bubbling** restores the heap-order property by swapping element $e$ along a downward path from the root.
• Downheap terminates when $e$ reaches a leaf or a node whose children have keys greater than or equal to $\text{key}(e)$.
• Since a heap has height $O(\log n)$, downheap runs in $O(\log n)$ time.

![Downheap diagram](image-url)
UPDATING THE LAST NODE

• The insertion node can be found by traversing a path of $O(\log n)$ nodes
  • Go up until a left child or the root is reached
  • If a left child is reached, go to the right child
  • Go down left until a leaf is reached

• Similar algorithm for updating the last node after a removal
VECTOR-BASED HEAP IMPLEMENTATION

- We can represent a heap with $n$ elements by means of a vector of length $n + 1$
  - Links between nodes are not explicitly stored
  - The leaves are not represented
  - The cell at index 0 is not used
- For the node at index $i$
  - the left child is at index $2i$
  - the right child is at index $2i + 1$
- $\text{insert}(e)$ corresponds to inserting at index $n + 1$
- $\text{removeMin}()$ corresponds to removing element at index $n$
- Yields in-place heap-sort
MERGING TWO HEAPS

- We are given two heaps and a new element $e$
- We create a new heap with a root node storing $e$ and with the two heaps as subtrees
- We perform downheap to restore the heap-order property
We can construct a heap storing $n$ given elements in using a bottom-up construction with $\log n$ phases.

In phase $i$, pairs of heaps with $2^i - 1$ elements are merged into heaps with $2^{i+1} - 1$ elements.
EXAMPLE
EXAMPLE
EXAMPLE
EXAMPLE
• We visualize the worst-case time of a downheap with a proxy path that goes first right and then repeatedly goes left until the bottom of the heap (this path may differ from the actual downheap path)

• Since each node is traversed by at most two proxy paths, the total number of nodes of the proxy paths is $O(n)$

• Thus, bottom-up heap construction runs in $O(n)$ time
## PRIORITY QUEUE SUMMARY

<table>
<thead>
<tr>
<th></th>
<th>insert(e)</th>
<th>removeMin()</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ordered List</td>
<td>(O(n))</td>
<td>(O(1))</td>
</tr>
<tr>
<td>Unordered List</td>
<td>(O(1))</td>
<td>(O(n))</td>
</tr>
<tr>
<td>Binary Heap, Vector-based Heap</td>
<td>(O(\log n))</td>
<td>(O(\log n))</td>
</tr>
</tbody>
</table>
One weakness of the priority queues so far is that we do not have an ability to update individual entries, like in a changing price market or bidding service.

We incorporate concept of positions to accomplish this (similar to List).

Additional ADT support (also includes standard priority queue functionality):

- **insert(e)** – insert element e into priority queue and return a position referring to this entry
- **remove(p)** – remove the entry referenced by position p
- **replace(p, e)** – replace with e the element associated with position p and return the position of the altered entry
- **Locators** decouple positions and entries in order to support efficient adaptable priority queue implementations (i.e., in a heap)
- Each position has an associated locator
- Each locator stores a pointer to its position and memory for the entry
• **Position**
  - represents a “place” in a data structure
  - related to other positions in the data structure (e.g., previous/next or parent/child)
  - often implemented as a pointer to a node or the index of an array cell
• **Position-based ADTs** (e.g., sequence and tree) are fundamental data storage schemes

• **Locator**
  - identifies and tracks a (key, element) item
  - unrelated to other locators in the data structure
  - often implemented as an object storing the item and its position in the underlying structure
• **Key-based ADTs** (e.g., priority queue) can be augmented with locator-based methods
We can use a priority queue to sort a set of comparable elements

Insert the elements one by one with a series of insert(e) operations

Remove the elements in sorted order with a series of removeMin() operations

Running time depends on the PQ implementation

Algorithm *PriorityQueueSort*()
Input: List *L* storing *n* elements and a Comparator *C*
Output: Sorted List *L*
1. Priority Queue *P* using comparator *C*
2. while ¬*L*.empty() do
   4.  *L*.eraseFront()
3. while ¬*P*.empty() do
   5.  *P*.removeMin()
7. return *L*
SELECTION-SORT

- Selection-sort is the variation of PQ-sort where the priority queue is implemented with an unsorted list.

- Running time of Selection-sort:
  - Inserting the elements into the priority queue with $n$ `insert(e)` operations takes $O(n)$ time.
  - Removing the elements in sorted order from the priority queue with $n$ `removeMin()` operations takes time proportional to

$$
\sum_{i=0}^{n} n - i = n + (n - 1) + \cdots + 2 + 1 = O(n^2)
$$

- Selection-sort runs in $O(n^2)$ time.
Selection-sort is the variation of PQ-sort where the priority queue is implemented with an unsorted list (do $n$ insert($e$) and then $n$ removeMin($e$)).

Illustrate the performance of selection-sort on the following input sequence:
- (22, 15, 36, 44, 10, 3, 9, 13, 29, 25)
Insertion-sort is the variation of PQ-sort where the priority queue is implemented with a sorted List

Running time of Insertion-sort:

- Inserting the elements into the priority queue with \( n \) \text{insert}(e) \) operations takes time proportional to
  \[
  \sum_{i=0}^{n} i = 1 + 2 + \cdots + n = O(n^2)
  \]

- Removing the elements in sorted order from the priority queue with a series of \( n \) \text{removeMin()} \) operations takes \( O(n) \) time

- Insertion-sort runs in \( O(n^2) \) time
EXERCISE
INSERTION-SORT

• Insertion-sort is the variation of PQ-sort where the priority queue is implemented with a sorted list (do $n$ insert($e$) and then $n$ removeMin($()$))

• Illustrate the performance of insertion-sort on the following input sequence:
  • (22, 15, 36, 44, 10, 3, 9, 13, 29, 25)
IN-PLACE INSERTION-SORT

• Instead of using an external data structure, we can implement selection-sort and insertion-sort in-place (only $O(1)$ extra storage)

• A portion of the input list itself serves as the priority queue

• For in-place insertion-sort
  • We keep sorted the initial portion of the list
  • We can use $\text{swap}(i, j)$ instead of modifying the list
Consider a priority queue with $n$ items implemented by means of a heap

- the space used is $O(n)$
- $\text{insert}(e)$ and $\text{removeMin}()$ take $O(\log n)$ time
- $\text{min}()$, $\text{size}()$, and $\text{empty}()$ take $O(1)$ time

Using a heap-based priority queue, we can sort a sequence of $n$ elements in $O(n \log n)$ time

The resulting algorithm is called heap-sort

Heap-sort is much faster than quadratic sorting algorithms, such as insertion-sort and selection-sort

Bottom-up heap construction is faster than $n$ successive insertions and speeds up the first phase of heap-sort
• Heap-sort is the variation of PQ-sort where the priority queue is implemented with a heap (do $n \text{ insert}(e)$ and then $n \text{ removeMin()}$)

• Illustrate the performance of heap-sort on the following input sequence (draw the heap at each step):
  • (22, 15, 36, 44, 10, 3, 9, 13, 29, 25)
# Priority Queue Summary

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<th>PQ-Sort total</th>
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<td>(O(n))</td>
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<td>Binary Heap, Vector-based Heap (Heap Sort)</td>
<td>(O(\log n))</td>
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