The running time of quicksort is proportional to the number of comparisons done in all the calls to PARTITION (comparing current array element $A[j]$ to the pivot $x$). To compute the expected running time of randomized quicksort, we need to compute the expected total number of comparisons done, over all the calls to PARTITION. We will refer to the smallest element in the input array $A$ as $z_1$, the second smallest element as $z_2$, etc.

Claim: Each pair of elements in $A$ is compared at most once.

Why? Elements are compared only to the pivot, and once a particular call to PARTITION finishes, that pivot is never compared to any element again.

We will use indicator random variables. Define

$$X_{ij} = \begin{cases} 1 & \text{if } z_i \text{ and } z_j \text{ are ever compared} \\ 0 & \text{otherwise} \end{cases}$$

where $0 \leq i \leq n - 1$ and $i + 1 \leq j \leq n$, i.e., subscript $j$ is always larger than subscript $i$.

The total number of comparisons, $X$, is a random variable with

$$X = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}$$

Now let’s calculate the expected value of $X$:

$$E[X] = E \left[ \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij} \right]$$

$$= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{ij}] \text{ by properties of expectation}$$

$$= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr[z_i \text{ and } z_j \text{ are compared}] \text{ by properties of indicator random variables}$$

So we need to calculate the probability that $z_i$ and $z_j$ are compared. Notice these key points:

- Once a pivot is chosen that is in between $z_i$ and $z_j$, $z_i$ and $z_j$ will be separated in the partition and will never be compared to each other.

- If $z_i$ is chosen as a pivot before any element in $\{z_{i+1}, z_{i+2}, \ldots, z_j\}$ is chosen as a pivot, then $z_i$ and $z_j$ will be compared.

- If $z_j$ is chosen as a pivot before any element in $\{z_i, z_{i+1}, \ldots, z_{j-1}\}$ is chosen as a pivot, then $z_i$ and $z_j$ will be compared.

So we need to calculate the probability that the first element in $\{z_1, \ldots, z_j\}$ chosen as a pivot is either $z_i$ or $z_j$.

Since pivots are chosen uniformly at random and independently, the two events ($z_i$ is chosen first, $z_j$ is chosen first) are independent.
\[
\text{Pr}[z_i \text{ and } z_j \text{ are compared}] = \text{Pr}[z_i \text{ or } z_j \text{ is first pivot chosen from } \{z_i, \ldots, z_j\}]
\]

\[
= \text{Pr}[z_i \text{ is first pivot chosen}] + \text{Pr}[z_j \text{ is first pivot chosen}]
\]

\[
= \frac{1}{j - i + 1} + \frac{1}{j - i + 1} \quad \text{since } \{z_i, \ldots, z_j\} \text{ contains } j - i + 1 \text{ elements}
\]

\[
= \frac{2}{j - i + 1}
\]

So we have:

\[
E[X] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j - i + 1}
\]

\[
= 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{1}{j - i + 1}
\]

\[
= 2 \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{1}{k + 1}
\]

\[
< 2 \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{1}{k}
\]

\[
= 2 \sum_{i=1}^{n} O(\log n) \quad \text{by Eq. A.7, bounding harmonic series}
\]

\[
= O(n \log n).
\]