Problem 30: Thrackles

CSCE620 Computational Geometry
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Problem 30 from The Open Problems Project
http://maven.smith.edu/~orourke/TOPP/
A thrackle is a drawing of a graph in the plane in which every two edges have precisely one point in common:
• if the two are incident, they share an end-point,
• otherwise, they share an interior point at which they cross each other
Thrackles in N-cycles

All cycles except $C_4$ are thrackleable
The Problem

What is the maximum number of edges in a thrackle?

Conway’s Conjecture
In any thrackle, the number of edges is at most equal to the number of vertices.
True or False?

Woodall, 1969

If the conjecture is true, then a graph is thrackleable iff it has at most one odd cycle and no cycle with exactly four edges.
According to Stephan Wehner, if the conjuncture is false, then theoretical counterexamples are:

1. two cycles connected by a path (called Dumb-bell)
2. three paths each connecting the same two points (called Theta)
3. two cycles sharing a vertex (called Figure-8)
Solved Case – Geometric Graphs

Hopf and Pannwitz, 1934; Sutherland, 1935

For any geometric thrackle $G$ on $n$ vertices, its number of edges is at most $n$.

What they were doing is to compute the maximum occurrences of the maximum distance among $n$ points in the plane.
History

1. László Lovász, János Pach, and Mario Szegedy. , 1995
   For thrackles on $n$ vertices, $m \leq 2n - 3$.

2. Cairns and Nikolayevsky, 2000
   For thrackles on $n$ vertices, $m \leq 1.5(n - 1)$.
Thank You
There have been some questions concerning the origin of the word thrackle. Conway says "When I was a teenager, on holiday with my parents in Scotland, we once stopped to ask directions of a man who was fishing by the side of a lake. He happened to mention that his line was thrackled. I'd previously called this kind of drawing a tangle, but since I'd just found a knot-theoretical use for that term, I changed this to thrackle. Several people have told me that they've searched in vain for this word in dialect dictionaries, but since I quizzed the fisherman about it, I'm sure I didn't mishear it; he really did use it." The name is indeed appropriate, as a drawing with the maximum number of crossings indeed resembles a highly-knotted fishing line.

(Taken from http://www.cems.uvm.edu/~archdeac/problems/thrackle.htm)
References