Edge-Coloring Geometric Graphs (Problem 75)

(http://maven.smith.edu/~orourke/TOPP/P75.html#Problem.75)
Statement

• For a set of n points in the plane in general position, draw a straight segment between every pair of points. What is the minimum number of colors that suffice to color the edges such that no two edges that cross have the same color?

• It has been conjectured that $(1 - \varepsilon)*n$ is an upper bound for some $\varepsilon > 0$. 
Approaching The Problem

• **Even Complete Graphs \([n = 2x]\)**
  • Does every such graph have a partition of its edge set into plane spanning trees?
  • The complete graph of \(n\) vertices has \(\frac{n(n-1)}{2}\) edges
  • A spanning tree has \((n-1)\) edges, so can have \(\frac{n}{2}\) trees (lower bound)
  • Then color each spanning tree a separate color

\[\text{n = 6, complete graph} \]
\[\text{3 spanning trees}\]
Approaching The Problem

- Convex Graphs – a geometric graph with the vertices in convex position
- Partition the graph into $\frac{n}{2}$ spanning caterpillar trees.
- A caterpillar is a tree such that deleting its leaves and leaf edges leaves a path.

Fig. 1. Partition of the convex $K_8$ into four spanning paths.
Approaching The Problem

- What about a *general* complete even graph?
- Sufficient Condition: partition graph into \( \frac{n}{2} \) spanning *double stars*.
- *Double Star* – A graph with at most two non-leaf vertices.
- *Proof* – in fig.8, assume a circle C and set of lines L, s.t. pts. of graph lie in the interior of C, s.t. only one pt. in each partition of C by L => can make spanning double star:

\[
T(V(G), L_i, i, i + \frac{n}{2})
\]

Fig. 8. Example of Theorem 9 with \( n = 4 \).
Generalizations

- Graphs with arbitrary no. of vertices
  - Using non-spanning trees
  - Every complete graph with $k$ crossings can be partitioned into $(n-k)$ plane trees. \[\text{[lemma-14, [2]]}\]
  - Aronov et al. [3] proved that every complete geometric graph $K_n$ has at least $\sqrt{n}/12$ pairwise crossing edges (called a crossing family).

  \[\Rightarrow (n - \sqrt{n}/12) \text{ plane trees}\]

- Is there an $\epsilon > 0$, such that every complete geometric graph $K_n$ can be partitioned into at most $(1 - \epsilon)n$ plane subgraphs?
References

