Regions and Other Concepts

Definition 11. Let G=(N,A,s) be a flow graph, let N₁⊆N, let A₁⊆A, and let h be in N₁. R=(N₁,A₁,h) is called a region of G with header h iff in every path (x₁, ..., xₖ), where x₁=s and xₖ is in N₁, there is some i ≤ k such that

(a) xᵢ = h
(b) xᵢ+1, ..., xₖ are in N₁
(c) (xᵢ,xᵢ₊₁), (xᵢ₊₁,xᵢ₊₂), ..., (xₖ₋₁,xₖ) are in A₁.

That is access to every node in the region is through the header only.

Lemma 12. A region of the flow graph is a subflowgraph.

Lemma 13. The header of a region dominates all nodes in the region.

Definition 12. We say that each node and arc in the original flow graph represents itself.

If T₁ is applied to node w with arc (w,w), then the resulting node represents what node w and arc (w,w) represented.

If T₂ is applied to x and y with arc (x,y) eliminated, then the resulting node z represents what x, y and (x,y) represented. In addition, if two arcs (x,u) and (y,u) are replaced by a single arc (z,u), the (z,u) represents what (x,u) and (y,u) represented.
Example:

Lemma 14. In a flow graph, if region R results from region R’ consuming region R”, then the header h of R’ dominates all nodes in R”.

4 represents \{2, b\}  e represents \{c, d\}

6 represents \{1, 2, 3, a, b, c, d\}
Theorem 6. As we reduce a flow graph G by T1 and T2, at all times the following conditions are true:

1. A node represents a region of G.
2. An edge from $x$ to $y$ represents a set of edges. Each such edge is from some node in the region represented by $x$ to the header of the region represented by $y$.
3. Each node and edge of G is represented by exactly one node or edge of the current graph.

Proof The theorem holds trivially for G itself. Every node is a region by itself, and every edge represents only itself.

Whenever T1 is applied, we add a self arc to a node representing a region. Adding the self arc does not change the fact that the node is a region.

Assume now that T2 is applied to consume node $y$ by node $x$. Let $x$ and $y$ represent regions $X$ and $Y$ respectively. Also, let A be the set of arcs represented by $(x,y)$. We claim that $X$, $Y$ and A together form a region whose header is the header of $X$. All we need is to prove that the header of $X$ dominates every node in $Y$. But this is true because T2 was applied and therefore all arcs entering $Y$ come from $X$. 
Definition 13. A parse $\pi$ of a reducible flow graph $G=(N,A,s)$ is a sequence of objects of the form $(T1,u,v,S)$ or $(T2,u,v,w,S)$, where $u$, $v$ and $w$ are nodes and $S$ is a set of arcs. We define the parse of a reducible flow graph recursively as follows:

1. The trivial flow graph has only the empty sequence as its parse.

2. If $G'$ (which may not be the original flow graph in a sequence of reductions) is reduced to $G''$ by an application of $T1$ to node $u$, and the resulting node is named $v$ in $G''$, then $(T1,u,v,S)$ followed by a parse of $G''$ is a parse of $G'$, where $S$ is the set of arcs represented by the arc $(u,u)$ eliminated from $G'$.

3. If $G'$ is reduced to $G''$ by an application of $T2$ to nodes $u$ and $v$ (with $u$ consuming $v$), and the resulting node is called $w$, then $(T2,u,v,w,S)$ followed by a parse of $G''$ is a parse of $G'$, where $S$ is the set of arcs represented by the arc $(u,v)$ in $G'$.

4. In both (2) and (3) above, “representation in $G'$ carries over to $G''$”. That is, whatever an object represents in $G'$ is also represented by that object in $G''$, except for those changes in representation caused by the particular transformation ($T1$ or $T2$) currently being applied.

Example: The parse of the previous flow graph is:

$$(T1,2,4,\{b\}) \ (T2,1,4,5,\{a\}) \ (T2,5,3,6,\{c,d\})$$
Definition 14. Let $G=(N,A,s)$ be a reducible flow graph and let $\pi$ be a parse of $G$. We say that an arc in $A$ is a back arc with respect to $\pi$ if it appears in set $S$ of an object $(T1,u,v,S)$ of $\pi$ and a forward arc (not to be confused with the forward arcs of a DFST. This is the forward arc of Definition 7’) with respect to $\pi$ otherwise. Let $B(G)$ be the set of arcs in $A$ that are back arcs in every parse of $G$.

Definition 15. A DAG of a flow graph $G=(N,A,s)$ is an acyclic flow graph $D=(N,A',s)$ such that $A'$ is a subset of $A$ and for any arc $e$ in $A-A'$, $(N,A'\cup\{e\},s)$ is not a DAG. That is, $D$ is a maximal acyclic subflowgraph.

Theorem 7. Let $G=(N,A,s)$ be a RFG and let $\pi$ be a parse of $G$. Arc $(x,y)$ is a back arc iff $y$ dominates $x$.

Theorem 8. A flow graph is reducible iff its DAG is unique.

Corollary The DAG of a reducible flowgraph is any DFST of $G$ plus its forward and cross arcs. (Alternatively, the back arcs of a parse of a reducible flow graph are exactly the back arcs of any DFST for $G$).

Corollary. See Definition 7’.
**Node Splitting**

How can an irreducible flow graph be transformed to an equivalent reducible flow graph?

First, let us assume that the nodes of a flow graph have (not necessarily distinct) labels. If $P=(x_1, \ldots, x_k)$ is a path in a flow graph, then we define $\text{labels}(P)$ to be the string of labels of these nodes: $(\text{label}(x_1), \ldots, \text{label}(x_k))$.

We say that two flow graphs $G_1$ and $G_2$ are equivalent iff, for each path $P$ in $G_1$, there is a path $Q$ in $G_2$ such that $\text{labels}(P)=\text{labels}(Q)$ and conversely.

Let $G=(N,A,s)$ be a flow graph. Let $x$ ($x \neq s$) be a node with no self loop, predecessors $w_1, \ldots, w_p$ ($p \geq 2$), and successors $y_1, \ldots, y_t$ ($t \geq 0$). We define $G\text{ split }x$ to be the flow graph resulting from the following procedure:

1. Delete the arcs from the $w_i$s to $x$ and those from $x$ to the $y_i$s.
2. Add $p$ copies of $x$ ($x_1, \ldots, x_p$) with $\text{label}(x_i) = \text{label}(x)$. Add arcs $(w_i,x_i)$ and arcs from every $x_i$ to all $y_j$s.

**Theorem 9.** Let $S$ denote the splitting of a node. Any flow graph can be transformed into the trivial flow graph by a transformation represented by the regular expression $T^\circ(ST^\circ)^*$. That is, first apply $T^\circ$, then apply $S$ followed by $T^\circ$ zero or more times.